Calculus @ QFinance

Lesson 2.2

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Differentiability. Maxima and Minima

professor Daniele Ritelli

www.unibo.it/docenti/daniele.ritelli

Existence of partial derivatives **does not ensure** continuity

In fact

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$$

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Differentiability. We define what it means for a vector function to be differentiable at a point. Whatever our definition, we expect two things: If f is differentiable at $\boldsymbol{a}$ then

(1) f will be continuous at $\boldsymbol{a}$

(2) all first-order partial derivatives of f will exist at $\boldsymbol{a}$.

Definition. Let f be a real function of n variables. f is said to be differentiable at a point $\boldsymbol{a} \in \mathbb{R}^n$ if and only if there is an open set V containing $\boldsymbol{a}$ such that $f : V \to \mathbb{R}$ and there is a $\boldsymbol{d} \in \mathbb{R}^n$ such that

$$\lim_{\boldsymbol{h} \to 0} \frac{f(\boldsymbol{a} + \boldsymbol{h}) - f(\boldsymbol{a}) - \boldsymbol{d} \cdot \boldsymbol{h}}{||\boldsymbol{h}||} = 0$$

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Theorem. If f is differentiable at $\boldsymbol{a}$, then:

(i) f is continuous at $\boldsymbol{a}$.

(ii) all first-order partial derivatives of f exist at $\boldsymbol{a}$.

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$\nabla f(\boldsymbol{a})$ is called the *gradient* or *nabla* of f in $\boldsymbol{a}$

To reverse the statement of the last Theorem if we strengthen the conclusion, we can obtain a reverse implication.

Theorem. Let V be open in $\mathbb{R}^n$, let $\boldsymbol{a} \in V$, and suppose that $f : V \to \mathbb{R}$. If all first-order partial derivatives of f exist in V and are continuous at $\boldsymbol{a}$, then f is differentiable at $\boldsymbol{a}$.

Chain Rule Suppose that $g = (g_1, \dots, g_n)$ is a vector function from $I \subseteq \mathbb{R}$ to $\mathbb{R}^n$, being I an open interval and $f : g(I) \subseteq \mathbb{R}^n \to \mathbb{R}$. If each of the components g_j of g is differentiable at $t_0 \in I$ and if f is differentiable at $\boldsymbol{a} = (g_1(t_0), \dots, g_n(t_0))$ then $\varphi(t) := f(g(t))$ is differentiable at t_0 and

$$\varphi'(t_0) = \nabla f(\boldsymbol{a}) \cdot g'(t_0)$$

where we set

$$g'(t_0) := (g_1'(t_0), \dots, g_n'(t_0))$$

and $\cdot$ is the dot (inner) product in $\mathbb{R}^n$

Jacobian matrix

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ a function from Euclidean n-space to Euclidean m-space. Such a function is given by m real-valued component functions, $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$. The partial derivatives of all these functions (if they exist) can be organized in an m-by-n matrix, the Jacobian matrix J of f, as follows:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} := \frac{\partial(f_1, \ldots f_m)}{\partial(x_1, \ldots, x_n)}$$

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The i^{th} row ($i = 1, \ldots, m$) of this matrix corresponds to the gradient of the i^{th}component function ∇f_i

Definition. (needed for what follows)

(i) For each $r > 0$, the open ball centered at $\boldsymbol{a}$ of radius r is the set of points

$$B_r(\boldsymbol{a}) := \{\boldsymbol{x} \in \mathbb{R}^n : \ ||\boldsymbol{x} - \boldsymbol{a}|| < r\}$$

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Notice that when $n = 1$, the open ball centered at a of radius r is the open interval $(a - r, a + r)$, and the corresponding closed ball is the closed interval $[a - r, a + r]$.

Critical points

Definition. Let V be an open set in $\mathbb{R}^n$, let $\boldsymbol{a} \in V$ and suppose that $f : V \to \mathbb{R}$.

(i) $f(\boldsymbol{a})$ is called a *local minimum* of f if and only if there is an $r > 0$ such that $f(\boldsymbol{a}) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in B_r(\boldsymbol{a})$

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(iii) $f(\boldsymbol{a})$ is called a *local extremum* of f if and only if $f(\boldsymbol{a})$ is a local maximum or a local minimum of f.

Remark. If the first-order partial derivatives of f exist at $\boldsymbol{a}$, and $f(\boldsymbol{a})$ is a local extremum of f, then $\nabla f(\boldsymbol{a}) = \mathbf{0}$.

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In fact the one-dimensional function

$$g(t) = f(a_1, \ldots, a_{j-1}, t, a_{j+1}, \ldots, a_n)$$

has a local extremum at $t = a_j$ for each $j = 1, \ldots, n$. Hence, by the one-dimensional theory

$$\frac{\partial f}{\partial x_j}(\boldsymbol{a}) = g'(a_j) = 0$$

Remark. If the first-order partial derivatives of f exist at $\boldsymbol{a}$, and $f(\boldsymbol{a})$ is a local extremum of f, then $\nabla f(\boldsymbol{a}) = \mathbf{0}$.

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As in the one-dimensional case, $\nabla f(\boldsymbol{a}) = \mathbf{0}$ is necessary but not sufficient for $f(\boldsymbol{a})$ to be a local extremum.

Remark. There exist continuously differentiable functions that satisfy $\nabla f(\boldsymbol{a}) = \mathbf{0}$ such that $f(\boldsymbol{a})$ is neither a local maximum nor a local minimum.

Consider for $n = 2$

$$f(x, y) = y^2 - x^2$$

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It is easy to check that $\nabla f(\mathbf{0}) = \mathbf{0}$ but the origin is a saddle point see figure

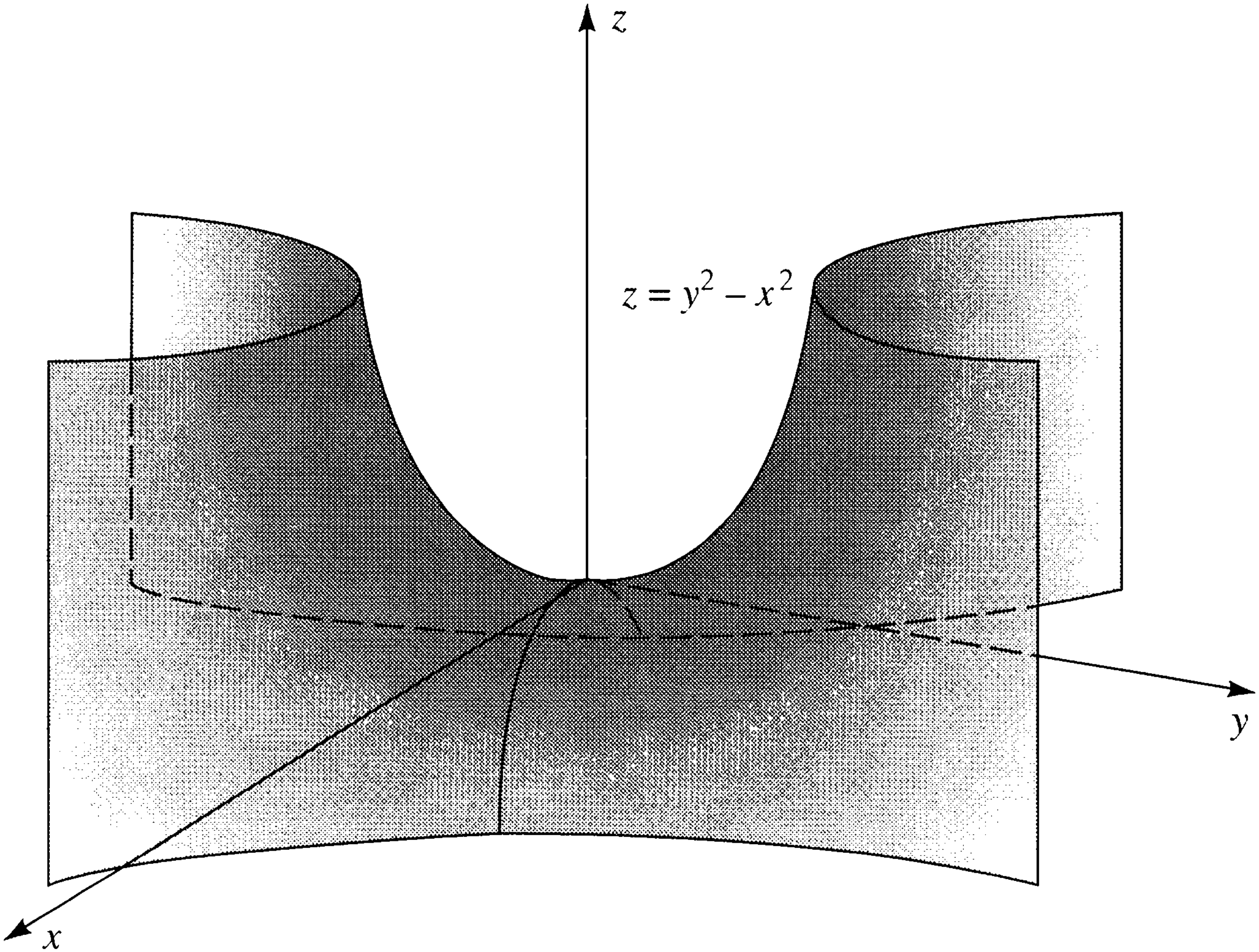

Figure 1: Saddle point

Definition. Let V be open in $\mathbb{R}^n$, let $\boldsymbol{a} \in V$, and let $f : V \to \mathbb{R}$ be differentiable at $\boldsymbol{a}$. Then $\boldsymbol{a}$ is called a *saddle point* of f if $\nabla f(\boldsymbol{a}) = \mathbf{0}$ and there is a $r_0 > 0$ such that given any $0 < \rho < r_0$ there are points $\boldsymbol{x}$, $\boldsymbol{y} \in B_\rho(\boldsymbol{a})$ that satisfy

$$f(\boldsymbol{x}) < f(\boldsymbol{a}) < f(\boldsymbol{y})$$

Hessian matrix. Let $V \subseteq \mathbb{R}^n$ an open set and let $f : V \to \mathbb{R}$ be a $\mathcal{C}^2$ function. The Hessian matrix of f at $\boldsymbol{x} \in V$ (or simply the Hessian) is the symmetric square matrix of second-order partial derivatives of f at $\boldsymbol{x}$:

$$H(f)(\boldsymbol{x}) := \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x})\right]_{i,j=1,\ldots,n}$$

Test for extrema and saddle points

Theorem. Let V be open in $\mathbb{R}^2$, $(a,b) \in V$, and suppose that $f : V \to \mathbb{R}$ satisfies $\nabla f(a,b) = \mathbf{0}$. Suppose further that $f \in \mathcal{C}^2$ and set

$$D := f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b)$$

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Observe that

$$D = \det[H(f)(a,b)]$$

Examples.

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$$\boldsymbol{f(x,y) = x^2 + y^3 - 2xy - y}$$

has a saddle point in $(a, b) = (-\frac{1}{3}, -\frac{1}{3})$ and a local minimum in $(a, b) = (1, 1)$

Remark. In n variables a critical point $\boldsymbol{x}_0$ is a local minimum for $f \in \mathcal{C}^2$ if for each $k = 1, \dots, n$

$$\det[H_k(f)(\boldsymbol{x}_0)] > 0$$

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Theorem (Lagrange multipliers) Let $m < n$, V be open in $\mathbb{R}^n$ and f, $g_j : V \to \mathbb{R}$ be $\mathcal{C}^1$ on V for $j = 1, 2, \ldots, m$. Suppose that rank of

$$\frac{\partial(g_1, \ldots g_m)}{\partial(x_1, \ldots, x_n)}$$

is m in $\boldsymbol{x}_0 \in V$ where $g_j(\boldsymbol{x}_0) = 0$ for $j = 1, 2, \ldots, m$ and suppose that $\boldsymbol{x}_0$ is a local extremum for f in the set

$$M = \{\boldsymbol{x} \in V : \; g_j(\boldsymbol{x}) = 0\}.$$

Then there exist scalars $\lambda_1, \ldots, \lambda_m$ such that

$$\nabla \left(f(\boldsymbol{x}_0) - \sum_{k=1}^{m} \lambda_k g(\boldsymbol{x}_0) \right) = \mathbf{0}$$

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Critical point equations

$$\begin{cases} -4y - 2\lambda x = 0, \\ -4x - 2\lambda y = 0, \\ x^2 + y^2 = 1. \end{cases}$$

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$$\lambda = 2 \implies \mathcal{P}_1 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \mathcal{P}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

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$$\lambda = -2 \implies \mathcal{P}_3 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \mathcal{P}_4 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$